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Massless Boundary Sine-Gordon Model Coupled to External Fields

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Abstract

We investigate a generalization of the massless boundary sine-Gordon model with conformal invariance, which has been used to describe an array of D-branes (or rolling tachyon). We consider a similar action whose couplings are replaced with external fields depending on the boundary coordinate. Even in the presence of the external fields, this model is still solvable, though it does not maintain the whole conformal symmetry. We obtain, to all orders in perturbation theory in terms of the external fields, a simpler expression of the boundary state and the disc partition function. As a by-product, we fix the relation between the bare couplings and the renormalized couplings which has been appeared in papers on tachyon lump and rolling tachyon.

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1 Introduction

Investigations of two dimensional quantum field theories play significant roles in several fields of physics. Especially, solvable models with boundary interactions find various applications in condensed matter physics (for example, [1]), string theory and purely theoretical interests. Boundary sine-Gordon model [2] is one of such models.

Recent years, in the field of string theory, the massless boundary sine-Gordon model with conformal invariance has been often used to represent an array of localized D-branes [3] or a time-dependent solution of string theory called rolling tachyon [4, 5]. (For a recent review of these topics, see [6].) The action of this model is given by[1]

$$S = \frac{1}{8\pi}\int d\sigma d\tau \partial_a X \partial^a X - \frac{1}{2}\int d\theta \left(g e^{\frac{iX(\theta)}{\sqrt{2}}} + \bar{g} e^{\frac{-iX(\theta)}{\sqrt{2}}} \right). \tag{1}$$

(For the case of rolling tachyon, the Wick-rotated version of this action is used. In this paper, we shall concentrate on the above one.) This model has been investigated in the papers [7, 8, 9, 10] in detail and proved to have conformal invariance. The boundary state for the boundary condition derived from this action was calculated in [8],

$$\langle B| = \langle N| e^{-i\pi(g_r J_0^+ + \bar{g}_r J_0^-)}, \tag{2}$$

where $\langle N|$ is the Neumann boundary state in the free theory at the self-dual radius, $R = \sqrt{2}$. J_0^+ and J_0^- are zero-modes of holomorphic $SU(2)$ currents. g_r and $\bar{g}_r$ are some renormalized couplings which should be determined by the original couplings, g and $\bar{g}$, appearing in the action (1). It is well-known that the absolute value of the renormalized couplings are in the range of $[-\frac{1}{2}, \frac{1}{2}]$. However, the explicit relation between these two kinds of couplings has not been given except for the case of some special limits.

In this paper, we consider a model with the following generalized boundary interaction,

$$S_{\text{int}} = -\frac{1}{2}\int d\theta \left[g(\theta) \exp\left(\frac{iX(\theta)}{\sqrt{2}}\right) + \bar{g}(\theta) \exp\left(\frac{-iX(\theta)}{\sqrt{2}}\right) \right], \tag{3}$$

where the external field $g(\theta)$ is an arbitrary function whose period is 2π and $\bar{g}(\theta)$ is the complex conjugate of $g(\theta)$. Due to the explicit coordinate dependence of the external field, this model does not have conformal symmetry. Nevertheless, we can define this theory as a conformal perturbation theory and solve it to all orders in perturbation theory. We calculate the boundary state for this theory with the help of differential equations. In this process, we can also fix the relation between the bare couplings and the renormalized ones in the original theory (2), mentioned above. Because the boundary state possesses the all information about the boundary interaction, we can easily calculate other quantities of this theory.

The outline of this paper is as follows. In section 2, we perturbatively evaluate the boundary state for the massless boundary sine-Gordon model with arbitrary external fields. In this calculation we shall fix our regularization scheme and notations. In section 3, we derive the boundary state to all orders in perturbation theory by introducing another technique, namely, solving differential equations. In section 4, we calculate the disc partition function and the correlation functions of this model, using the results of section 3, and mention a symmetry of this model. In section 5, we summarize and discuss our results in this paper.

[1] In this paper, we take closed string picture, $0 \leq \sigma \leq 2\pi$. The boundary lies at $\tau = 0$, where we conventionally describe the coordinate σ as θ. And we use the convention of $\alpha' = 2$.

2 Regularization and Some Perturbative Calculations

In this section, we shall apply the technique used in [8] to our model (3) and obtain some perturbative results for its boundary state. Though the application is straightforward, we shall provide this section for fixing our conventions and regularization scheme.

In order to make the following calculations well-defined, we slightly shift the positions of the boundary interactions to the inside of the disc by a short distance ϵ,

$$: \exp\left(\frac{\pm iX(\theta,-\epsilon)}{\sqrt{2}}\right) :, \tag{4}$$

This distance ϵ is the parameter for our regularization. Note that, in order to define the above operators, we have used a "bulk" normal ordering where the divergences from image charges are not subtracted. Therefore, they do not coincide with the original operators on the boundary, even in the limit of $\epsilon \to 0$. In the limit of $g(\theta) \to 0$, the relation between these two different normal orderings should be given by

$$O[X(\theta)] = \lim_{\epsilon\to 0} \exp\left(\frac{1}{2}\log|2\epsilon|^2 \left(\frac{\partial}{\partial X}\right)^2\right) : O[X(\theta,-\epsilon)] : . \tag{5}$$

The left-hand operator is defined by the boundary normal ordering where effects of image charges are taken into account. Therefore, for the interaction terms of our model, we should multiply the regularized operators (4) by a divergent factor, $1/\sqrt{2\epsilon}$. This regularization is essentially the same one as in the original paper [8].

The boundary state $\langle B|$ is defined by acting the exponential of the boundary interaction (3) on the Neumann boundary state $\langle N|$,

$$\langle B| = \langle N| \exp\left(\frac{1}{2}\int d\theta \left[\frac{g(\theta)}{\sqrt{2\epsilon}}\exp\left(\frac{iX(\theta,-\epsilon)}{\sqrt{2}}\right) + \frac{\bar{g}(\theta)}{\sqrt{2\epsilon}}\exp\left(\frac{-iX(\theta,-\epsilon)}{\sqrt{2}}\right)\right]\right). \tag{6}$$

Following the argument in [8], we compactify the target space X into the self-dual radius, $X \sim X + 2\pi\sqrt{2}$, and perturbatively simplify this state using the $SU(2)$ current algebra. In the $O(g(\theta))$ term, we have

$$\begin{aligned}
&\frac{1}{2}\langle N| \int d\theta \frac{g(\theta)}{\sqrt{2\epsilon}} : e^{i\frac{X(\theta,-\epsilon)}{\sqrt{2}}} : \\
&= \frac{1}{2}\langle N| \int d\theta \frac{g(\theta)}{\sqrt{2\epsilon}} : e^{i\frac{X_R(\theta,-\epsilon)}{\sqrt{2}}} :: e^{i\frac{X_L(\theta,-\epsilon)}{\sqrt{2}}} : \\
&= \frac{1}{2}\langle N| \int d\theta e^{-\frac{\pi i}{4}} \frac{g(\theta)}{\sqrt{2\epsilon}} : e^{i\frac{X_L(\theta,\epsilon)}{\sqrt{2}}} :: e^{i\frac{X_L(\theta,-\epsilon)}{\sqrt{2}}} : \\
&= -i\pi\langle N| \oint \frac{dz}{2\pi i} g(z) : e^{i\sqrt{2}X_L(z)} : .
\end{aligned} \tag{7}$$

In the second line of (7), we have separated the operator into holomorphic and anti-holomorphic parts. The second equality comes from a property of the Neumann boundary state, namely $\langle N|X_R(\theta,-\tau) = \langle N|X_L(\theta,\tau)$. The phase $e^{\frac{\pi i}{4}}$ is due to an effect of the normal ordering[2] and it cancels another phase

[2] There is an ambiguity related to the choice of branches. Here, we just pick up the one for reality of the boundary state.

from the OPE among two holomorphic operators. In the last line, we have used a holomorphic coordinate $z = e^{-i\sigma+\tau}$, which is nothing but $e^{-i\theta}$ on the boundary. We have also rewritten the external field in terms of z. The full expression for the $O(g(\theta),\bar{g}(\theta))$ terms is given by,

$$-i\pi\langle N|\oint \frac{dz}{2\pi i}\left(g(z)j^{+}(z)+\bar{g}(z)j^{-}(z)\right), \tag{8}$$

where $j^{+}(z) =: e^{i\sqrt{2}X_L(z)}:$ and $j^{-}(z) =: e^{-i\sqrt{2}X_L(z)}:$. Note that this final expression does not contain divergent quantities any more.

Next, we shall consider the second-order terms,

$$\langle N|\frac{1}{2!}S_{\rm int}S_{\rm int} = -\frac{i\pi}{2!}\langle N|\oint \frac{dz}{2\pi i}\left(g(z)j^{+}(z)+\bar{g}(z)j^{-}(z)\right)\frac{1}{2}\int d\theta\left[\frac{g(\theta)}{\sqrt{2\epsilon}}:\exp\left(\frac{iX(\theta,-\epsilon)}{\sqrt{2}}\right):+\text{c.c.}\right]. \tag{9}$$

Because anti-holomorphic operators commute with holomorphic charges, we can repeat the same argument as in (7) for these terms. For the first half of these terms, we have

$$\begin{aligned}-\frac{i\pi}{2!}\langle N|\frac{1}{2}\int d\theta\frac{e^{-\frac{\pi i}{4}}}{\sqrt{2\epsilon}}\Bigg[g(\theta)\oint \frac{dz}{2\pi i}\left(g(z)j^{+}(z)+\bar{g}(z)j^{-}(z)\right):e^{i\frac{X_L(\theta,\epsilon)}{\sqrt{2}}}::e^{i\frac{X_L(\theta,-\epsilon)}{\sqrt{2}}}:\\ -g(\theta)\bar{g}(\theta):e^{-i\frac{X_L(\theta,\epsilon)}{\sqrt{2}}}::e^{i\frac{X_L(\theta,-\epsilon)}{\sqrt{2}}}:\Bigg].\end{aligned} \tag{10}$$

The main difference from the previous calculations is that, we have an OPE between operators with the opposite charges. This kind of OPE brings some divergent terms as follows,

$$\frac{(i\pi)^2}{2!}\langle N|\oint \frac{dw}{2\pi i}\left[g(w)j^{+}(w)\oint \frac{dz}{2\pi i}\left(g(z)j^{+}(z)+\bar{g}(z)j^{-}(z)\right)-g(w)\bar{g}(w)(\frac{1}{2\epsilon w}+j^3(w))\right], \tag{11}$$

where $j^3(w)$ is one of the $SU(2)$ currents defined by $i\partial X(w)/\sqrt{2}$. The whole expression for the second-order terms is given by,

$$\langle N|\frac{1}{2!}\left[\left(-i\pi\oint \frac{dz}{2\pi i}\left(g(z)j^{+}(z)+\bar{g}(z)j^{-}(z)\right)\right)^2-2(i\pi)^2(2\epsilon)^{-1}\int\frac{d\theta}{2\pi}g(\theta)\bar{g}(\theta)\right]. \tag{12}$$

While the linear term of $j^3(w)$ in (11) has been canceled out by its complex conjugate, we still have a divergent term.

In principle, we can continue these arguments and obtain, order by order, the expression of the boundary state, which is represented by $SU(2)$ charges. However, as the order becomes higher, the expression becomes more complex and the divergences become worse. For example, for the third order terms, we have

$$\begin{aligned}\langle N|\frac{1}{3!}\Bigg[\left(-i\pi\oint \frac{dz}{2\pi i}\left(g(z)j^{+}(z)+\bar{g}(z)j^{-}(z)\right)\right)^3+(-i\pi)^3\oint\frac{dz}{2\pi i}\left(gg\bar{g}(z)j^{+}(z)+\bar{g}\bar{g}g(z)j^{-}(z)\right)\\ -6(-i\pi)^3(2\epsilon)^{-1}\int\frac{d\theta}{2\pi}g(\theta)\bar{g}(\theta)\oint\frac{dz}{2\pi i}\left(g(z)j^{+}(z)+\bar{g}(z)j^{-}(z)\right)\Bigg].\end{aligned} \tag{13}$$

In the next section, we shall derive the whole expression of this boundary state to all orders in perturbation theory using some differential equations and explicitly show that we can remove the divergences by introducing only one counter term corresponding to a constant shift of the boundary potential.

3 Boundary State to All Orders in Perturbation Theory

In the original work [8] on the conformal invariant theory (1), it is proved that the above divergences are absorbed as a constant shift in the potential, and terms in the perturbation series are exponentiated and take the following simple form,

$$\langle B| = \langle N| e^{-i\pi(g_r J_0^+ + \bar{g}_r J_0^-)}, \tag{14}$$

where g_r and $\bar{g}_r$ are renormalized couplings which should be non-trivial functions of original couplings g and $\bar{g}$. However, in their work, the explicit dependence of the constant shift and the renormalized couplings on the original couplings has not been given. Though the constant shift of the potential does not affect the physics of this two dimensional theory, it should correspond to a condensation of the spatially homogeneous tachyon in the string-theory's point of view. Thus, it seem to be worth while to investigate it in more detail. Furthermore, in this paper, we consider the generalized theory with the coordinate-dependent interaction (3). Then, the relation between "couplings" becomes more important, because "couplings" are not just constant but some functions of θ, and the relation can contain more information.

In order to obtain the expression for the boundary state of our generalized model to all orders of original couplings $g(\theta)$ and $\bar{g}(\theta)$, we shall introduce a new technique here. The previous result (14) and our perturbative results in the previous section suggest that the boundary state in terms of original couplings is given by

$$\langle B| = \langle N| \exp\left[\int d\theta \left(\frac{1}{2}\alpha(g(\theta),\bar{g}(\theta))e^{i\sqrt{2}X_L(\theta)} + \frac{1}{2}\bar{\alpha}(g(\theta),\bar{g}(\theta))e^{-i\sqrt{2}X_L(\theta)} + \beta(g(\theta),\bar{g}(\theta))\right)\right]. \tag{15}$$

The functions, $\alpha(\theta)$ and $\bar{\alpha}(\theta)$, are some functions which are fixed by the original external fields, $g(\theta)$ and $\bar{g}(\theta)$. They are reduced to the renormalized couplings, g_r and $\bar{g}_r$, when external fields are just constants. We have added a function, $\beta(g,\bar{g})$, in order to represent possible constant shifts of the potential. Our strategy is to find differential equations which our functions, $\alpha(g,\bar{g})$, $\bar{\alpha}(g,\bar{g})$ and $\beta(g,\bar{g})$, should satisfy by comparing the two expressions of the derivative of the boundary state. Using these differential equations, we can obtain the exact expression for the boundary state.

First, we start from the original definition (6) of the boundary state. By differentiating it with respect to the external field, $g(\theta)$, we have

$$\frac{\delta}{\delta g(\theta)}\langle B| = \langle B|\frac{1}{2\sqrt{2\epsilon}}\int_0^1 ds e^{\frac{-s}{2}\int d\theta\left(\frac{g}{\sqrt{2\epsilon}}e^{\frac{iX}{\sqrt{2}}} + \frac{\bar{g}}{\sqrt{2\epsilon}}e^{\frac{-iX}{\sqrt{2}}}\right)} e^{\frac{iX(\theta)}{\sqrt{2}}} e^{\frac{s}{2}\int d\theta\left(\frac{g}{\sqrt{2\epsilon}}e^{\frac{iX}{\sqrt{2}}} + \frac{\bar{g}}{\sqrt{2\epsilon}}e^{\frac{-iX}{\sqrt{2}}}\right)}. \tag{16}$$

Especially, the following combination of the derivatives takes a simple form:

$$\int d\theta\left(g(\theta)\frac{\delta}{\delta g(\theta)} + \bar{g}(\theta)\frac{\delta}{\delta \bar{g}(\theta)}\right)\langle B| = \langle B|\frac{1}{2}\int d\theta\left(\frac{g(\theta)}{\sqrt{2\epsilon}}e^{\frac{iX(\theta)}{\sqrt{2}}} + \frac{\bar{g}(\theta)}{\sqrt{2\epsilon}}e^{\frac{-iX(\theta)}{\sqrt{2}}}\right). \tag{17}$$

In the r.h.s of the above expression, we can replace the original definition for $\langle B|$ with the ansatz (15). Because the holomorphic operators, $e^{\pm\frac{iX_L}{\sqrt{2}}}$, behave as spin 1/2 states for the holomorphic $SU(2)_L$ charges, using the following equation,

$$\exp\left(-i\pi\begin{pmatrix}0 & \alpha\\ \bar{\alpha} & 0\end{pmatrix}\right) = \begin{pmatrix}\cos(\pi\sqrt{\alpha\bar{\alpha}}) & -i\sqrt{\alpha/\bar{\alpha}}\sin(\pi\sqrt{\alpha\bar{\alpha}})\\ -i\sqrt{\bar{\alpha}/\alpha}\sin(\pi\sqrt{\alpha\bar{\alpha}}) & \cos(\pi\sqrt{\alpha\bar{\alpha}})\end{pmatrix}, \tag{18}$$

we can easily repeat the same arguments made in the previous section. Thus, the combination of the derivatives of the boundary state is given by

$$
\begin{aligned}
-i\pi\langle B|\oint\frac{dz}{2\pi i}\Bigg(&g\cos(\pi\sqrt{\alpha\bar{\alpha}})j^{+}(z)+\bar{g}\cos(\pi\sqrt{\alpha\bar{\alpha}})j^{-}(z)\\
&-i\sin(\pi\sqrt{\alpha\bar{\alpha}})\left(\frac{1}{2\epsilon z}\left(g\sqrt{\bar{\alpha}/\alpha}+\bar{g}\sqrt{\alpha/\bar{\alpha}}\right)+(g\sqrt{\bar{\alpha}/\alpha}-\bar{g}\sqrt{\alpha/\bar{\alpha}})j^{3}(z)\right)\Bigg).
\end{aligned}
\tag{19}
$$

We assume that the two functions $\alpha(g,\bar{g})$ and $\bar{\alpha}(g,\bar{g})$ are related by a single function $f(x)$ which depends only on the absolute value of $g(\theta)$, $x(\theta)=|g(\theta)|$, namely,

$$
\alpha(g,\bar{g})=gf(x),\qquad \bar{\alpha}(g,\bar{g})=\bar{g}f(x),
\tag{20}
$$

so that the last term of (19) vanishes. Next, we shall evaluate the same combination of the derivatives of the boundary state by differentiating the ansatz (15) directly. The above assumption (20) about the dependence of the functions α and $\bar{\alpha}$ also makes this evaluation quite simple. The quantity at issue is evaluated as follows:

$$
\begin{aligned}
&-i\pi\langle B|\int_0^1 ds e^{i\pi s\oint\frac{dz}{2\pi i}(\alpha j^{+}+\bar{\alpha}j^{-})}\oint\frac{dz}{2\pi i}\left(g\frac{d(xf)}{dx}j^{+}+\bar{g}\frac{d(xf)}{dx}j^{-}+\frac{2i}{z}x\frac{d\beta}{dx}\right)e^{-i\pi s\oint\frac{dz}{2\pi i}(\alpha j^{+}+\bar{\alpha}j^{-})}\\
&=-i\pi\langle B|\oint\frac{dz}{2\pi i}\left(g\frac{d(xf)}{dx}j^{+}(z)+\bar{g}\frac{d(xf)}{dx}j^{-}(z)+\frac{2i}{z}x\frac{d\beta}{dx}+i\pi\frac{\partial x}{\partial z}x\left(\frac{d(xf)}{dx}\right)^2\right).
\end{aligned}
\tag{21}
$$

Here, we have also assumed that the function $\beta(g,\bar{g})$ depends only on $x(\theta)$. Comparing these two expressions for the derivative of the boundary state, we have following differential equations

$$
\frac{d(xf(x))}{dx}=\cos(\pi xf(x)),
\tag{22}
$$

$$
\frac{d\beta(x)}{dx}=\frac{1}{2\epsilon}\sin(\pi xf(x))-\frac{\pi z}{2}\frac{\partial x}{\partial z}\left(\frac{d(xf)}{dx}\right)^2.
\tag{23}
$$

Using these two differential equations, we can fix the functions, $\alpha(\theta)$, $\bar{\alpha}(\theta)$ and $\beta(\theta)$, in terms of the external fields, $g(\theta)$ and $\bar{g}(\theta)$. With the suitable initial condition, the solution of (22) is given by

$$
f(x)=\frac{2}{\pi x}\arctan\left[\tanh\left(\frac{\pi}{2}x\right)\right].
\tag{24}
$$

Thus, we can determine the functions, $\alpha(\theta)$ and $\bar{\alpha}(\theta)$, which appear in the exponent of the boundary state, as

$$
\alpha(g(z),\bar{g}(z))=\frac{2}{\pi}\frac{g(z)}{|g(z)|}\arctan\left[\tanh(\frac{\pi}{2}|g(z)|)\right],
\tag{25}
$$

$$
\bar{\alpha}(g(z),\bar{g}(z))=\frac{2}{\pi}\frac{\bar{g}(z)}{|g(z)|}\arctan\left[\tanh(\frac{\pi}{2}|g(z)|)\right].
\tag{26}
$$

This relation is one of our main results in this paper. If we turn off the non-zero modes of the external fields, $g(z)$ and $\bar{g}(z)$, these equations give the relation between the renormalized couplings, g_r and $\bar{g}_r$, and the bare ones, g and $\bar{g}$:

$$
g_r=\frac{2g}{\pi|g|}\arctan\left[\tanh(\frac{\pi}{2}|g|)\right],
\tag{27}
$$

$$
\bar{g}_r=\frac{2\bar{g}}{\pi|g|}\arctan\left[\tanh(\frac{\pi}{2}|g|)\right].
\tag{28}
$$

In the small coupling limit, $|g| \to 0$, g_r approaches g. On the contrary, in the large coupling limit, $|g| \to \infty$, the renormalized coupling, $|g_r|$, approaches $1/2$. This property is exactly what the authors of [8] has conjectured for the renormalized couplings as a finite renormalization effect. They have shown that the boundary state (14) represents the Dirichlet boundary condition when its renormalized coupling $|g_r|$ is $1/2$. On the other hand, since, in the large coupling limit, the interaction potential forces the boundary value of the field to be fixed, the corresponding boundary state should be the Dirichlet boundary state. It means that the renormalized coupling should approach $1/2$ in this limit. Our results (27) and (28) explicitly describe this renormalization effect.

Next, we consider the equation (23) for constant shifts of the boundary potential. It is convenient to split the function β as $\beta = \beta_1 + \beta_2$, and to consider the following two equations:

$$
\begin{aligned}
\frac{d\beta_1(x)}{dx} =& \frac{1}{2\epsilon}\sin(\pi x f(x)), & (29)\\
\frac{d\beta_2(x)}{dx} =& -\frac{\pi z}{2}\frac{\partial x}{\partial z}\left(\frac{d(xf)}{dx}\right)^2 = -\frac{z}{2}\frac{\partial}{\partial z}\sin(\pi x f(x)). & (30)
\end{aligned}
$$

The integrations of them give

$$
\begin{aligned}
\beta_1(g(z), \bar{g}(z)) =& \frac{1}{2\pi\epsilon}\log(\cosh(\pi|g(z)|)), & (31)\\
\beta_2(g(z), \bar{g}(z)) =& -\frac{z}{2\pi}\frac{\partial}{\partial z}\log(\cosh(\pi|g(z)|)). & (32)
\end{aligned}
$$

The function β_1 depends on the regularization parameter ϵ and diverges in the limit of $\epsilon \to 0$. However, this divergence is absorbed by a counter-term, or a constant shift of the boundary interaction and it is harmless, as expected. Furthermore, the function β_2 affects nothing, because it gives a integration of a total derivative. From the view point of string theory, the function β corresponds to the spatially homogeneous tachyon. It tells us how much we should add the spatially homogeneous tachyon in addition to the space-dependent one. Equivalently, this information can be absorbed into a definition of the boundary operators. For example,

$$
e^{iX(\theta)/\sqrt{2}} \equiv \lim_{\epsilon\to 0}\left[\frac{1}{\sqrt{2\epsilon}} : e^{iX(\theta,-\epsilon)/\sqrt{2}} : -\frac{1}{2\pi\epsilon g(\theta)}\log(\cosh(\pi|g(\theta)|))\right], \tag{33}
$$

and a similar definition for the operator $e^{-iX(\theta)/\sqrt{2}}$. Under such a prescription, our final expression for the boundary state for the theory with the interaction (3) is

$$
\begin{aligned}
\langle B| =& \langle N|\exp\left[-2i\oint\frac{dz}{2\pi i}\left(\frac{g(z)}{|g(z)|}\arctan\left(\tanh(\frac{\pi}{2}|g(z)|)\right)j^+(z)\right.\right.\\
&\left.\left.+\frac{\bar{g}(z)}{|g(z)|}\arctan\left(\tanh(\frac{\pi}{2}|g(z)|)\right)j^-(z)\right)\right]. \qquad (34)
\end{aligned}
$$

Strictly speaking, our procedure is not strong enough to determine this boundary state completely, except for the case that $g(\theta)$ is just real constant g. Our differential equations are nothing but consistency conditions on our ansatz (15). However, we have checked that our result (34) correctly reproduces the perturbative results in the previous section to at least first few orders. Therefore, we just conjecture that our result (34) is the correct one even with external fields, $g(\theta)$ and $\bar{g}(\theta)$.

In fact, we can directly check whether the state (34) satisfies the boundary condition derived from the action with external fields. The boundary condition is given by

$$\frac{1}{\sqrt{2}\pi}\frac{dX(\theta)}{d\tau} - ig(\theta)e^{i\frac{X(\theta)}{\sqrt{2}}} + i\bar{g}(\theta)e^{-i\frac{X(\theta)}{\sqrt{2}}} = 0. \tag{35}$$

However, there is an ambiguity related to the definition of the boundary operators. Because this boundary condition is for the interacting theory, the boundary operators appearing in the above condition must be the operators which are defined in this interacting theory. We prefer to define these operators using differentiation with respect to the external fields. Under this prescription, the boundary condition for the boundary state is represented as

$$\langle B|\frac{-1}{\sqrt{2}\pi}\frac{dX(\theta)}{d\tau} = 2\left(ig(\theta)\frac{\delta}{\delta g(\theta)} - i\bar{g}(\theta)\frac{\delta}{\delta\bar{g}(\theta)}\right)\langle B|. \tag{36}$$

A short calculation leads the both sides of (36) equally into the following form:

$$\begin{aligned}
&-\frac{z}{\pi}\langle B|\left(\frac{1}{2}\sqrt{\frac{\alpha}{\bar{\alpha}}}\sin(2\pi\sqrt{\alpha\bar{\alpha}})j^{+}(z) - \frac{1}{2}\sqrt{\frac{\bar{\alpha}}{\alpha}}\sin(2\pi\sqrt{\alpha\bar{\alpha}})j^{-}(z)\right.\\
&\qquad\left. - 2i\sin^2(\pi\sqrt{\alpha\bar{\alpha}})j^3(z) + \frac{i}{2}\sin^2(\pi\sqrt{\alpha\bar{\alpha}})\left(\frac{\partial\bar{\alpha}}{\bar{\alpha}} - \frac{\partial\alpha}{\alpha}\right)\right).
\end{aligned} \tag{37}$$

Thus, our "boundary state" actually satisfies the boundary condition. Note that, in this calculation, we have no need to use the concrete relations (27) and (28) between external fields and renormalized ones. Therefore, we cannot fix their relations from this condition.

4 Disc Partition Function and Correlation Functions

In this section, we shall consider the disc partition function and some correlation functions[3] of this model. To calculate the disc partition function $Z[g(\theta)] = \langle B|0\rangle$, we must reexpress the boundary state only in terms of non-negative modes. In the case that $g(\theta)$ is a real function, we can easily do it using our expression (34). In the rest of this section, we shall focus on such a case. Then, the boundary state can be rewritten as

$$\begin{aligned}
\langle B| =& \langle N|\exp\left(-2\pi i\sum_{n=-\infty}^{\infty}\alpha_n J^1_n\right)\\
=& \langle N|\exp\left(2\pi i\sum_{n=1}^{\infty}\alpha_{-n}\tilde{J}^1_n\right)\exp\left(-2\pi i\sum_{n=1}^{\infty}\alpha_n J^1_n\right)\exp\left(i\pi\alpha_0\left(\tilde{J}^1_0 - J^1_0\right) - 4\pi^2\sum_{n=1}^{\infty}n\alpha_n\alpha_{-n}\right),
\end{aligned} \tag{38}$$

where $J^1_n(\tilde{J}^1_n)$ are charges of holomorphic(anti-holomorphic) $SU(2)$ symmetry and the coefficients α_n are Fourier coefficients of the function $\alpha(\theta)$, namely, $\alpha(\theta) = \sum_n \alpha_n e^{-in\theta}$. Using the free energy,

$$w[g(\theta)] = 4\pi^2\sum_{n=1}^{\infty}n\alpha_n\alpha_{-n} = 4\pi^2\sum_{n=1}^{\infty}n\int\frac{d\theta}{2\pi}\int\frac{d\theta'}{2\pi}\alpha(\theta)\alpha(\theta')\cos n(\theta'-\theta), \tag{39}$$

[3] More general correlation functions of the original model (1) has been discussed in [8, 12].

the disc partition function of this system is written as

$$Z[g(\theta)]_{\text{Disc}} = \langle B|0\rangle = \exp\left(-w[g(\theta)]\right), \tag{40}$$

where we have simply normalized the inner product $\langle N|0\rangle$, which is independent of $g(z)$, as 1. If the external field is just constant, the partition function (or the free energy) does not depend on the coupling g. In on-shell case, we can regard a partition function as an action for a string field theory [11]. The independence of the partition function from the constant coupling g just tells us that, at the self-dual radius, the total energy of the D-brane does not change during the tachyon condensation into the lower dimensional D-brane [3]. Note that, for this independence, we must carefully select the tachyon profile (33).

The 1-point function of the boundary operator is directly calculated from the derivative of the free energy:

$$\left\langle \cos\left(\frac{X(\theta)}{\sqrt{2}}\right)\right\rangle = -\frac{\delta w}{\delta g(\theta)} = -4\pi^2\cos(\pi\alpha(\theta))\sum_{n=1}^{\infty} n\int\frac{d\theta'}{2\pi}\alpha(\theta')\cos n(\theta'-\theta). \tag{41}$$

This 1-point function vanishes if we take the external field as a constant. The connected 2-point function is given by

$$\begin{aligned}\left\langle \cos\left(\frac{X(\theta_1)}{\sqrt{2}}\right)\cos\left(\frac{X(\theta_2)}{\sqrt{2}}\right)\right\rangle_{\text{connected}} =& 4\pi^2\cos(\pi\alpha(\theta_1))\cos(\pi\alpha(\theta_2))\frac{1}{\sin^2(\frac{\theta_1-\theta_2}{2})}\\ &+\pi\delta(\theta_1-\theta_2)\sin(\pi\alpha(\theta_1))\left\langle\cos\left(\frac{X(\theta_1)}{\sqrt{2}}\right)\right\rangle. \end{aligned}\tag{42}$$

These correlation functions agree with the usual ones in the free limit of $g(\theta)\to 0$. In this limit, 1-point function just vanishes and 2-point function becomes[4]

$$\left\langle \cos\left(\frac{X(\theta_1)}{\sqrt{2}}\right)\cos\left(\frac{X(\theta_2)}{\sqrt{2}}\right)\right\rangle_{\text{free}} \sim \frac{1}{\sin^2(\frac{\theta_1-\theta_2}{2})}, \tag{43}$$

as expected. On the other hand, in the large coupling limit of $g\to\infty$ of the constant coupling, they cease to propagate, where $\cos(\pi\alpha) = 0$.

Before ending this section, we shall report a curious property of the partition function (40). The Lagrangian for our model has an explicit dependence on the boundary coordinate through the external field $g(\theta)$. Therefore, this model looks like a model which does not have conformal invariance except for the case that $g(\theta)$ is a constant. However, our disc partition function still knows the conformal invariance in the following sense. Let us consider the $SL(2,R)$ transformations of the external field, which map the boundary of the disk into itself. The infinitesimal transformation of the external field is given by

$$\delta g(z) = (\epsilon_0+\epsilon_1 z+\epsilon_2 z^2)\partial g(z), \tag{44}$$

where parameters satisfy $\bar\epsilon_0 = -\epsilon_2, \bar\epsilon_1 = -\epsilon_1$. It is easily shown that these transformations of the external field do not change the partition function (40). This fact suggests that, though our theory does not possesses the full symmetry under the conformal transformation, it still has a symmetry under the subgroup of the transformation, namely, $SL(2,R)$.

[4]Note that these arguments are rather formal. We have freely used integrations by parts to derive the 2-point function (42). We should evaluate well-defined quantities with integrations over the coordinate.

5 Summary and Discussion

In this paper, we have extended the argument by [8] into the case where their couplings are replaced with arbitrary periodic functions. Even in this generalized theory, their argument to rewrite the state in terms only of holomorphic fields still works. Using differential equations, we have obtained the boundary state, to all orders in perturbation theory, with the relations between the “bare external fields” coupled to the boundary interaction and the “renormalized external fields” which form charges with $SU(2)$ currents. This explicit determination of the boundary state for the generalized theory (3) is our main result. Using this boundary state, we have also calculated the disc partition function (or the free energy) of this system.

So far we have solved this theory, to all orders in perturbation theory, in spite of the existence of the boundary interaction (3). However, this is not so surprising. The free energy $w[g(\theta)]$ of this system is written by a quadratic form of $\alpha(\theta)$. This fact suggests that there exists a free theory coupled to the external field, not $g(\theta)$, but $\alpha(\theta)$, which becomes equivalent to our theory after path integration. Actually, such a theory exists for the original theory (1). Using fermionization, the authors of [9] mapped the theory into a fermionic system with quadratic interactions. Our results in this paper give a precise relation between these two theories.

In the presence of external fields, the corresponding boundary state does not satisfy reparametrization invariance conditions, $\langle B|(L_n - \tilde{L}_{-n}) = 0$. Thus, this state should be an off-shell state, though it satisfies the boundary condition (35). Although, at present, we have no specific applications of this model to the field of string theory, it would be interesting to find such candidates. Besides, in the original paper [8], the theory with conformal symmetry was studied in detail as a quantum field theory of two dimensional space-time. Similar kinds of analyses would be applicable to our generalized one.

Or rather, our generalized theory might be a suitable tool for investigations into the original one with conformal invariance. For example, we can simply define the local boundary operators of the interaction theory from the derivatives with respect to corresponding external fields without difficulties of divergences. And, there, all quantities are written in terms of ones appearing in the action (1). Though these results are rather technical, they might help us to study more important physics such as rolling tachyon.

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